Read-once Algebraic Branching Programs and Commuting Matrices

And why one should attend random talks

Anamay Tengse [with C Ramya (IMSc) and Vishwas Bhargava (Waterloo)]

Reichman University (IDC Herzliya)

An interesting question

Sums of powers of linear forms

- [Waring, 1770]: Let $k \in \mathbb{N}$, is there always a **finite** $g(k)$ such that any **positive integer** is a sum of k^{th} powers of at most $g(k)$ many integers?

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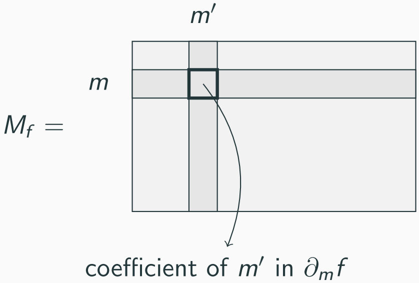
Q. Are there explicit polynomials with Waring rank $\exp(n)$?

Partial Derivatives

[Nisan & Wigderson 1996]:

Partial Derivative Matrix for f



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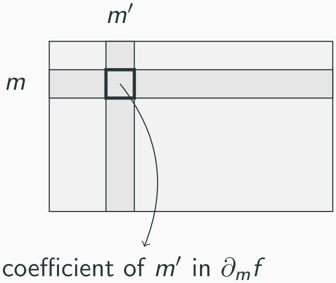
$M_f =$ 

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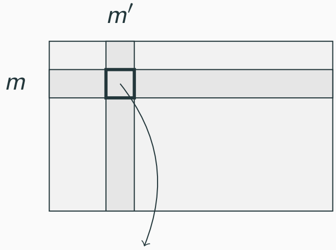

coefficient of m' in $\partial_m f$

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  - $\mathrm{WR}(x_1 x_2 \cdots x_n) = 2^{\Theta(n)}$.

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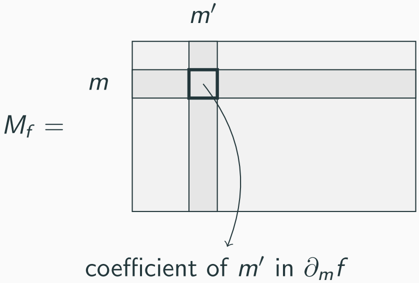

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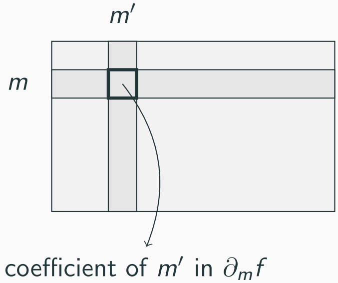

Question. Let $\mathrm{DPD}(f) = \mathrm{rk}(M_f)$. If $\mathrm{DPD}(f) \leq s$, is $\mathrm{WR}(f) \leq \mathrm{poly}(n, s)$?

Waring rank vs dimension of partial derivatives

- No known candidates for separating DPD and WR

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  - $\mathrm{DPD}(\mathrm{Det}_n) \leq 2^{2n}$, and $\mathrm{WR}(\mathrm{Det}_n) \leq 2^{O(n \log n)}$

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Question. Any other property separating WR and DPD?

Read-once Branching Programs

Waring rank and ROABPs

[Saxena 2008]: For any $\overline{a} \in \mathbb{C}^n$, $(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^d$ can be expressed as a sum of $O(nd)$ products of univariate polynomials.

$$(a_1x_1 + \cdots + a_nx_n)^d = \sum_{i \in [t]} g_{i,1}(x_1) \cdot g_{i,2}(x_2) \cdots g_{i,n}(x_n), \text{ for } t \leq n(d+1).$$

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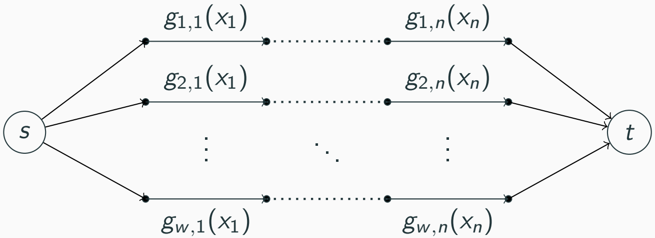
Question. What happens when $\mathrm{DPD}(f) = r$?

Sums of products of univariates, and ROABPs

$$\sum_{i=1}^{w}\left(\prod_{j=1}^{n} g_{i,j}(x_j)\right) \to$$



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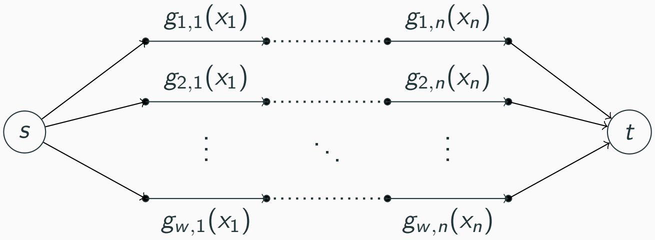

$$\bar{1}^{\intercal} \begin{bmatrix} D_1(x_1) \end{bmatrix} \cdots \begin{bmatrix} D_n(x_n) \end{bmatrix} \bar{1}$$

Here, $D_i(x_i)$ is **diagonal** $w \times w$ matrix with univariates in x_i.

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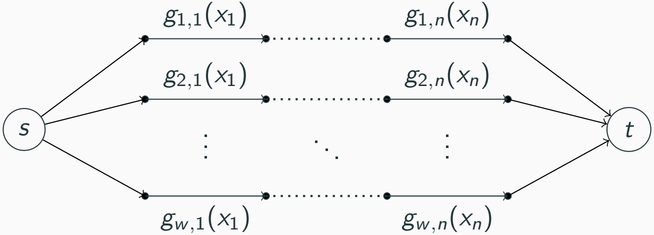

$$\bar{u}^{\mathsf{T}} \begin{bmatrix} \quad M_1(x_1) \quad \end{bmatrix} \cdots \begin{bmatrix} \quad M_n(x_n) \quad \end{bmatrix} \bar{v}$$

Here, $M_i(x_i)$ is **any** $w \times w$ matrix with univariates in x_i.

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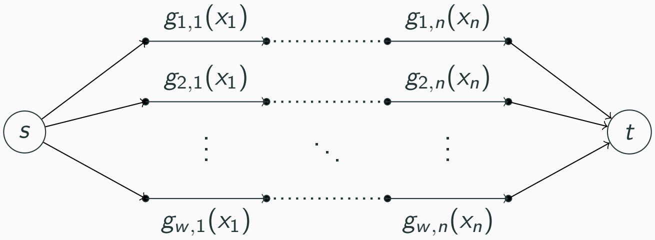

ROABP of width w.
Read-once,
Oblivious ABP.

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ROABPs and partial derivatives

- ROABP of width w, for f: $f(\overline{x}) = \bar{u}^{\intercal} \cdot M_1(x_1) \cdot M_2(x_2) \cdots M_n(x_n) \cdot \bar{v}$, where $M_i(x_i)$s are $w \times w$ matrices with univariates in x_i, $\bar{u}, \bar{v} \in \mathbb{C}^w$.

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- Order of the variables:
  Consider $g(\overline{x}, \overline{y}) = (x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$.
  Width required for g in the order $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, is 2.
  But in the order $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$, g requires width $\exp(n)$.

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- Fact. If $\mathrm{DPD}(f) \leq s$, then f has an ROABP of width s in **every order**.

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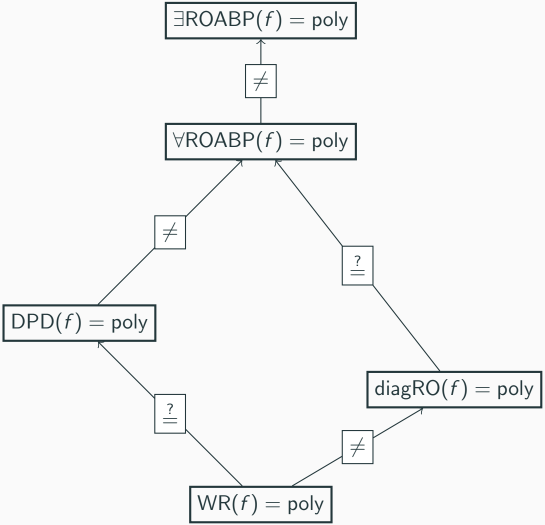
Question. What can ROABPs tell us about the DPD vs WR question?

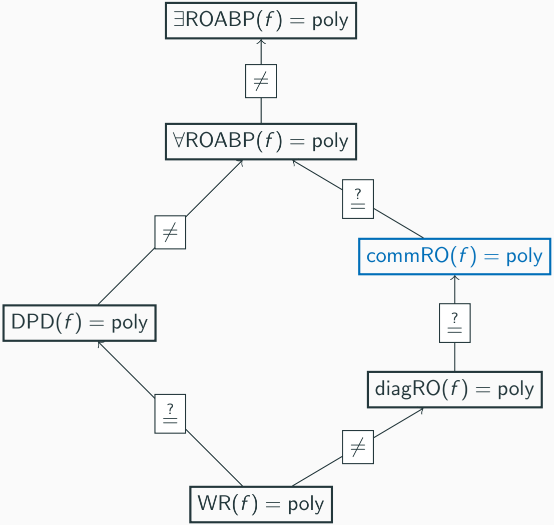
Our work

Family Portrait (old)



- $\exists\text{ROABP}(f) = \text{poly}$
  small ROABPs in **some** order
- $\forall\text{ROABP}(f) = \text{poly}$
  small ROABPs in **every** order
- $\text{diagRO}(f) = \text{poly}$
  small diagonal ROABPs
- $\text{WR}(f) = \text{poly}$
  small Waring rank
- $\text{DPD}(f) = \text{poly}$
  small dimension of partials

Family Portrait (old)

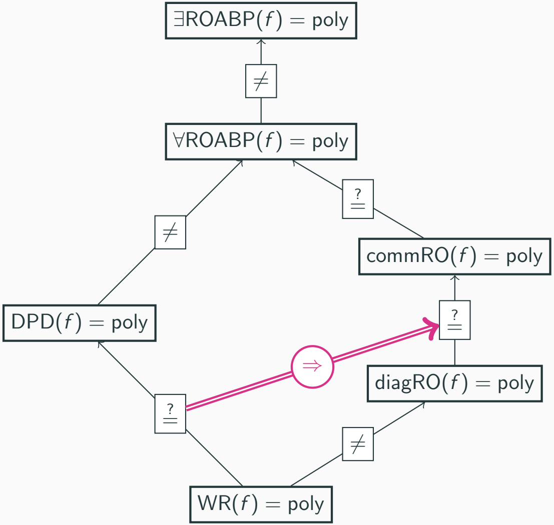


Commutative ROABPs

ROABPs with matrices that **pairwise commute** with each other.

- commRO(f) = poly
  small commutative ROABPs

Family Portrait (new)



Theorem 1 [Ramya-T. 2022]
If $\forall g, \mathrm{WR}(g) \leq (n \cdot \mathrm{DPD}(g))^a$, then
$\forall f, \mathrm{diagRO}(f) \leq O(n \cdot (\mathrm{commRO}(f))^{10a})$.

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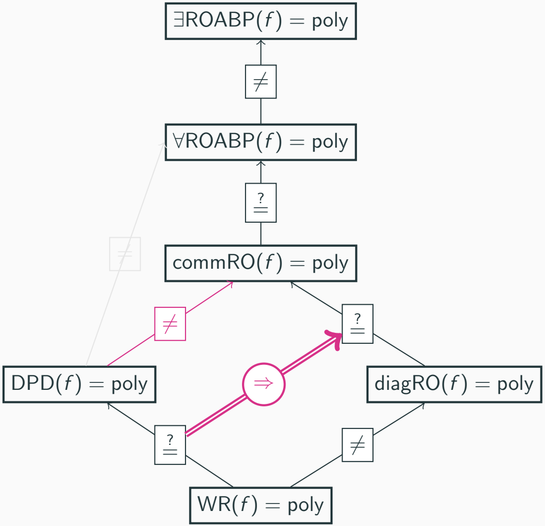

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Theorem 2 [*Bhargava-T. 2024*]

For any polynomial f, $\mathsf{commRO}(f) \leq O(\deg(f)^2 \cdot \mathsf{DPD}(f))$.

Key proof ideas

Ben-Or's trick

Elementary Symmetric Polynomial:

$$\mathrm{ESym}_n^d(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_d \leq n} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_d}$$

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[Ben-Or]: $\mathrm{ESym}_n^d(\overline{x}) = \mathrm{coeff}_{t^d}\left((1 + tx_1)(1 + tx_2) \cdots (1 + tx_n)\right)$. Thus,

$$\mathrm{ESym}_n^d(\overline{x}) = \sum_{j \in [n+1]} \beta_j \cdot (1 + jx_1)(1 + jx_2) \cdots (1 + jx_n), \text{ for some } \beta_1, \ldots, \beta_{n+1} \in \mathbb{C}$$

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Corollary. DiagRO for ESym_n^d of width $O(n)$ for any d.

CommRO for ESym

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$$\prod_{i=1}^{n}(I + x_i A) = \begin{bmatrix} 1 & \mathrm{ESym}_n^1 & \mathrm{ESym}_n^2 & \cdots & \mathrm{ESym}_n^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

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- $\mathrm{ESym}_n^d(\overline{x}) = \left(\prod_i (I + x_i A) \right)_{1,d+1}$
   - CommRO of width $O(d)$

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  - CommRO of width $O(d)$
- Setting $t = A$ is like going modulo t^{d+1}
  - Minimal polynomial of A: t^{d+1}

(Very) High Level Overview of Theorem 1

$$(1) \qquad \mathrm{ESym}_n^d(\overline{x}) = \mathrm{coeff}_{t^d}\left((1+tx_1)\cdots(1+tx_n)\right)$$

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Proof sketch

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- (1) — (3) with poly(n, w) blow-up, if $\mathrm{WR}(g) \leq \mathrm{poly}(n, \mathrm{DPD}(g))$ for all g [Pratt19]

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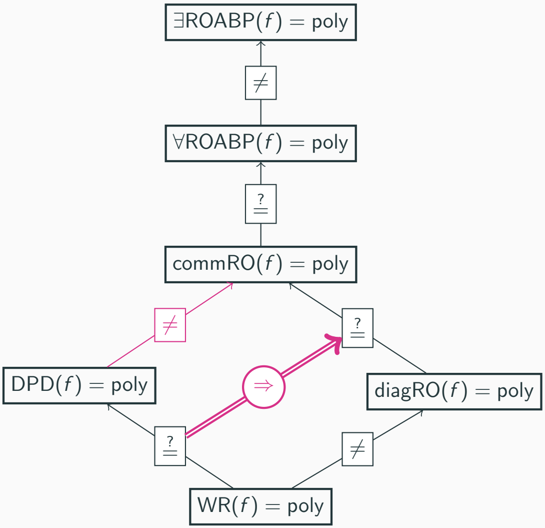
- Picking the matrices (like A for t^{d+1})
  - Fact. $\exists A_1, \ldots, A_n \in \mathbb{C}^{w \times w}$ corresponding to $f^\perp$, where $w = \text{DPD}(f)$.
  - $\exists \bar{v}$ such that $f(\overline{x}) = \text{firstRow}(G(A_1, \ldots, A_n, x_1, \ldots, x_n)) \cdot \bar{v}$.

Concluding remarks

Open questions



- Resolve any of the $\stackrel{?}{=}$ questions.
- Is the converse of theorem 1 true?

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- Moral of the story. If you're not busy, attend the talk.

Thank you!