Finding the Order of an ROABP

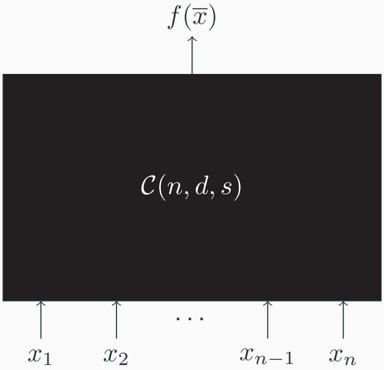
Can we proper-learn ROABPs?

Anamay Tengse

TIFR, June 11, 2025

joint work with Vishwas Bhargava, Pranjal Dutta and Sumanta Ghosh

Polynomial Identity Testing (PIT)



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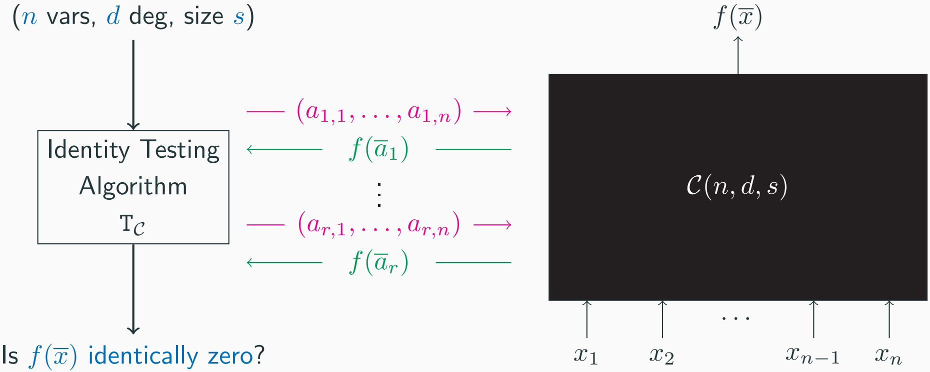

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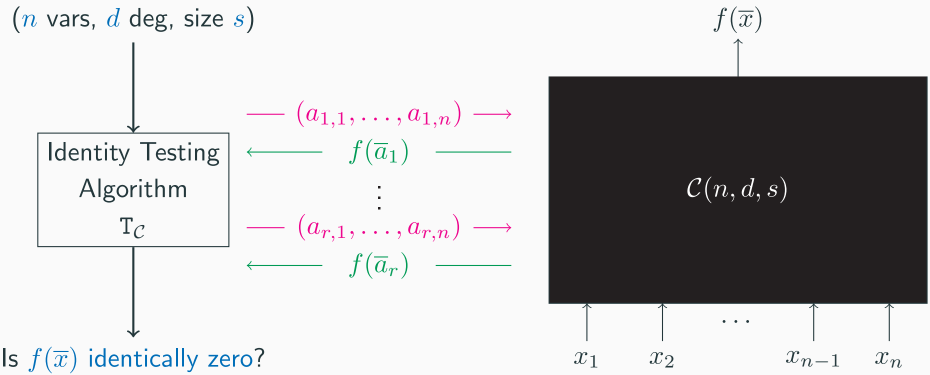
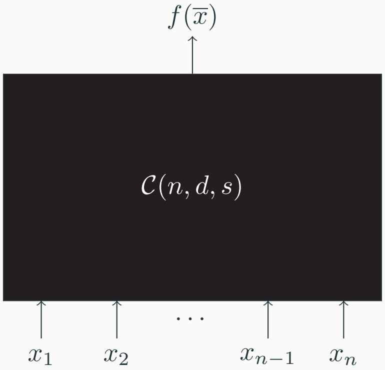


Known. Randomized, $\text{poly}(n, d, s)$-time algorithm for circuits.

Open. Deterministic algorithm for circuits, making fewer than d^n queries.

Reconstruction ("Proper-Learning") of Polynomials



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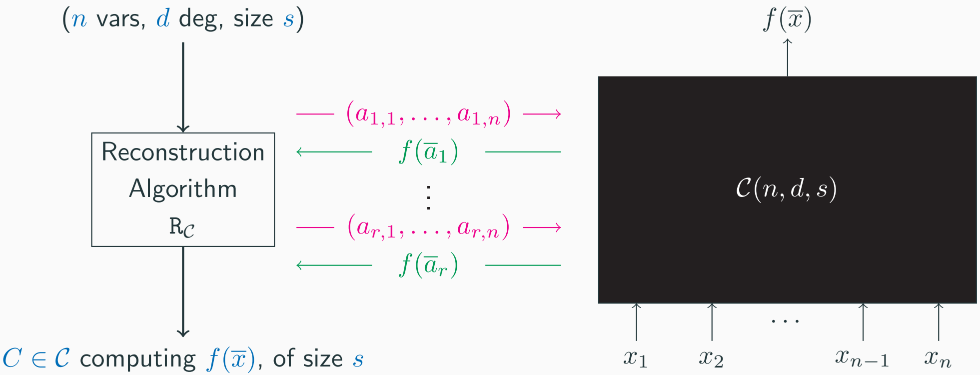

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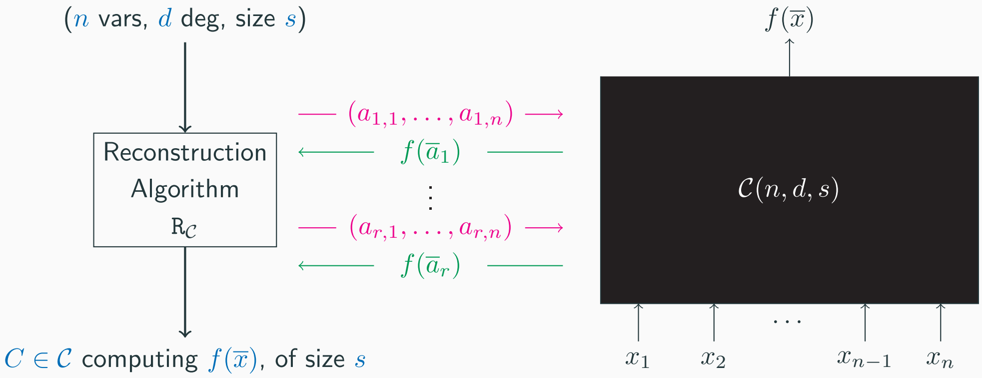
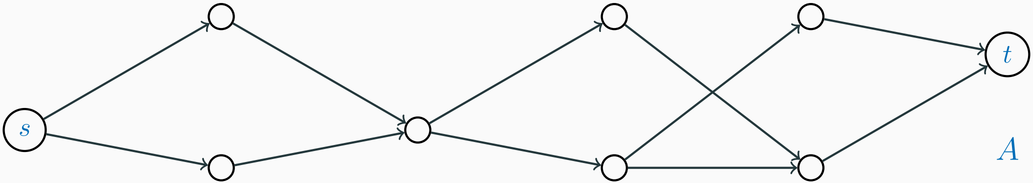


$\mathrm{R}_{\mathcal{C}}$ exists: "The class $\mathcal{C}$ can be proper-learnt in `<complexity-of-R>`"

The Question

Read-once Oblivious ABPs

Algebraic Branching Programs (ABPs)



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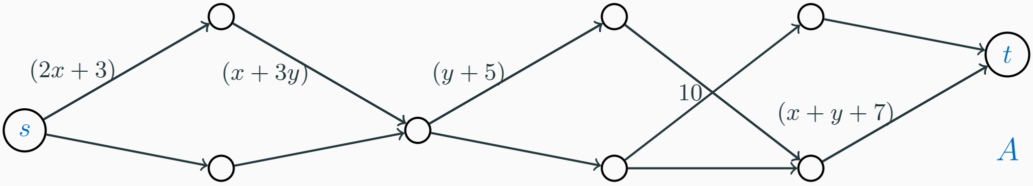

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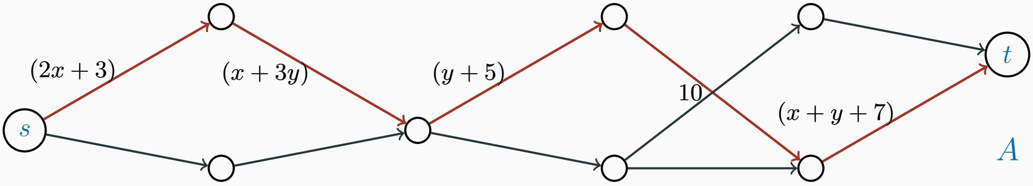

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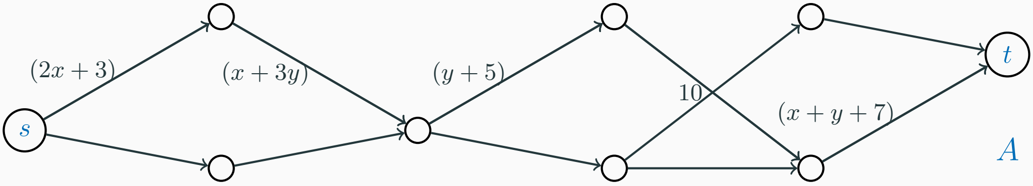

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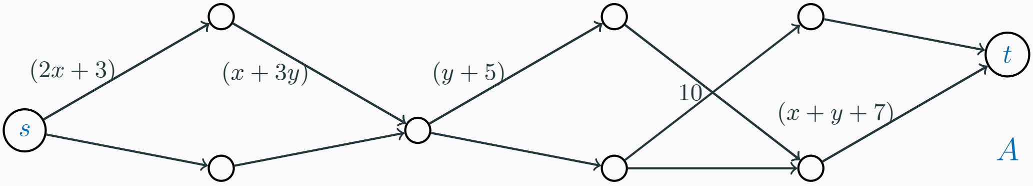

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- Size of the ABP: total number of vertices (9 in the example)

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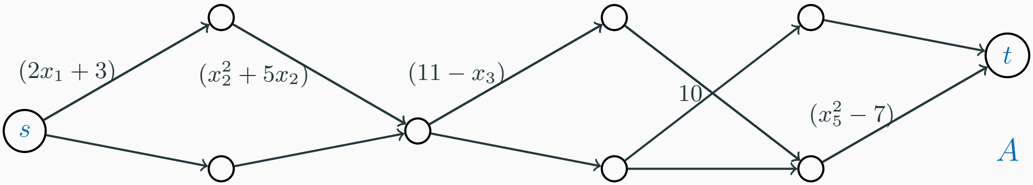

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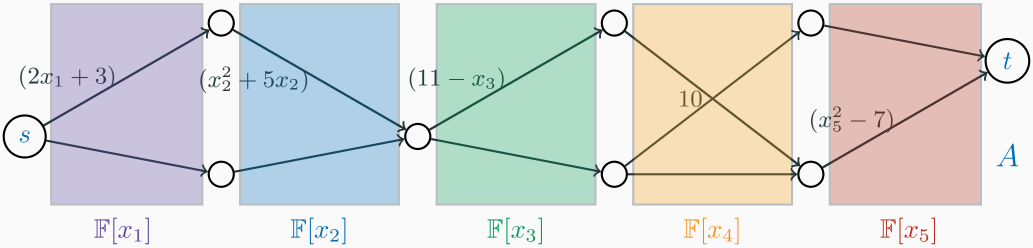

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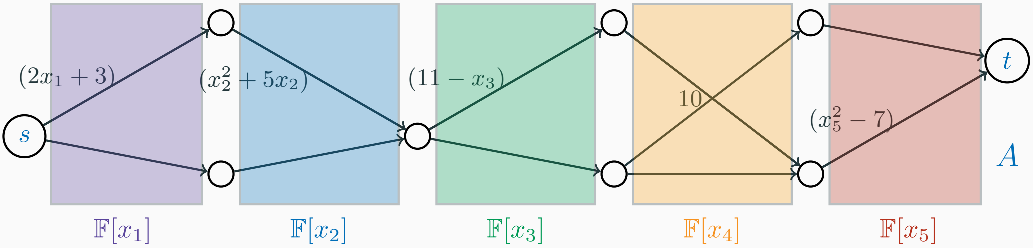

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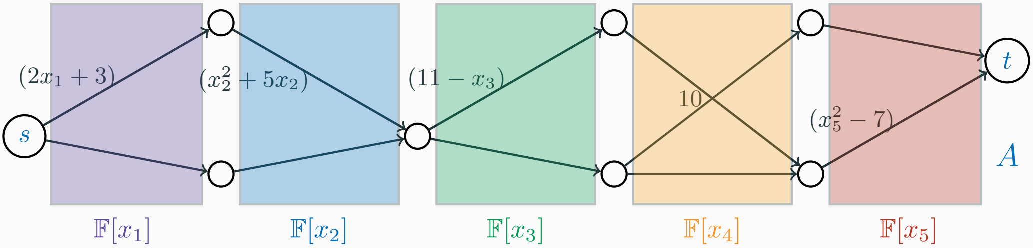

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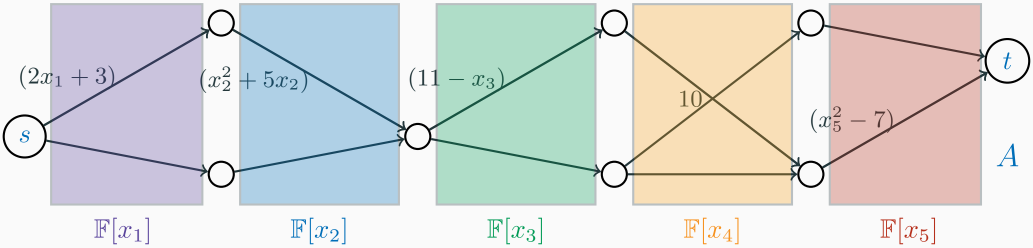

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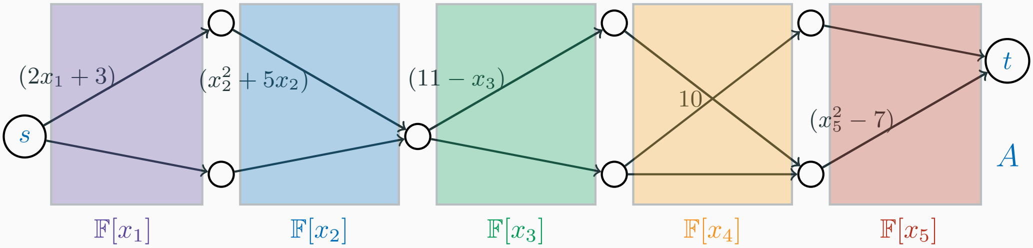

ROABP (n-variate, degree-d, width-w, order σ)

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- The i^{th} layer of edges only has degree-d univariates in $x_{\sigma(i)}$ as labels.
- Width of the ROABP: Maximum number of vertices in any layer.
- Order of the ROABP: permutation $\sigma \in s_n$ in which the variables are read.

Example. Width depends on Order

$F(\overline{x}, \overline{y})$ has a width 2 ROABP in the order $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$,
but requires width 2^n in the order $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$.

$$F(\overline{x}, \overline{y}) := (x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$$

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Q. What is the complexity of finding the order?

The Question

Order Finding Problem

Given parameters $n, d, w \in \mathbb{N}$ and a polynomial $f(\overline{x})$, find some order σ in which f has an ROABP of width at most w.

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Order Finding Problem (Decision)

Given an n-variate, degree-d polynomial $f(\overline{x})$, and a parameter $w \in \mathbb{N}$, determine if f has an ROABP of width at most w in some order σ.

Our Results

NP hardness (Algebraic Circuit Minimization)

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Average-case algorithm

Randomized order-finding algorithm that runs in polytime for a random/generic ROABP.

Proof Ideas

NP-hardness

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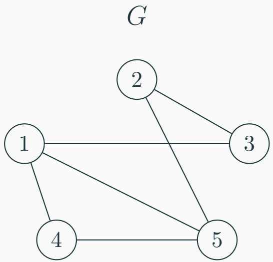

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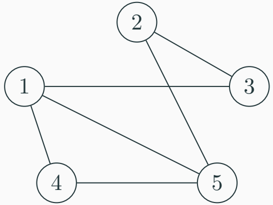

Arrangement of G with $(5, 2, 3, 1, 4)$

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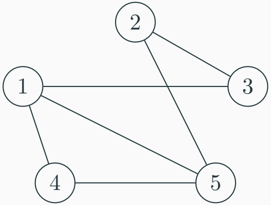

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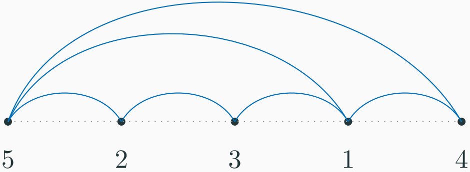

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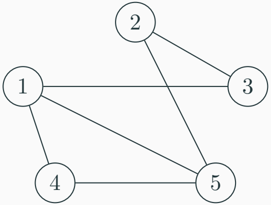

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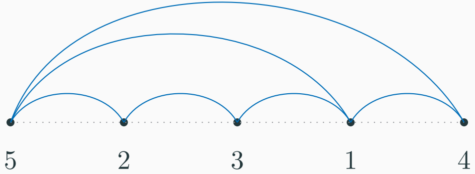

$$\text{CutWidth}(G) \;:=\; \min_{\sigma \in S_n} \text{CutWidth}_\sigma(G) \;:=\; \min_{\sigma \in S_n} \left(\max_{i \in [n]} \#\text{edges}(\sigma[1:i], \sigma[i+1:n]) \right)$$

Linear Arrangement of Graphs

Goal. Show that finding an optimal order is hard.

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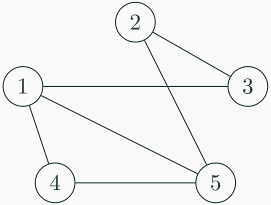
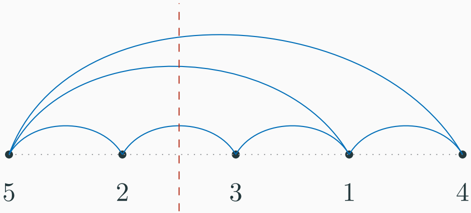


CutWidth of G for $(5, 2, 3, 1, 4)$ is 3



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$$\forall m \in \mathrm{mons}(\overline{x}_L), m' \in \mathrm{mons}(\overline{x}_R), \qquad M_f^{(\sigma,i)}[m, m'] = \mathrm{coeff}_f(m \cdot m')$$

Reduction from `CutWidth`

Lemma (Bhargava-Dutta-Ghosh-T. 2024)

Given any graph $G = (V, E)$, there is a polynomial $f_G(x_1, \ldots, x_n)$ such that:

- $n = |V|$,
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Fact (Monien-Sudborough 1988)

CutWidth is NP-complete, even for planar graphs of degree 3.

Theorem (Algebraic MCSP)

For any constant $\Delta \geq 6$, order finding for n-variate, degree-Δ polynomials is NP-hard, even when f is given in the dense representation (algebraic analogue of a truth table).

Proof. Truth table has length $\binom{n+\Delta}{\Delta} = \mathrm{poly}(n)$ for constant Δ.

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Theorem (Algebraic Circuit Minimization)

Order finding problem is NP-hard, even when f is given as an algebraic circuit.

Proof. $\mathrm{CircuitSize}(f_G) = O(n^3)$. (Truth table length $\sim 2^n$ for degree $\Omega(n)$.)

Proof Ideas

E-time worst-case algorithm

How to beat $n!$?

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E.g. For $n = 5$, $\sigma = (5,2,3,1,4)$, $\mathrm{rk}(M_f^{\{5\}})$, $\mathrm{rk}(M_f^{\{2,5\}})$, $\mathrm{rk}(M_f^{\{2,3,5\}})$ and $\mathrm{rk}(M_f^{\{1,2,3,5\}})$ are all at most w.

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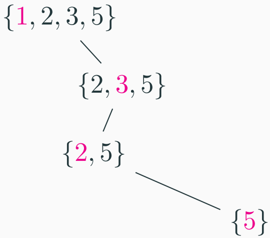

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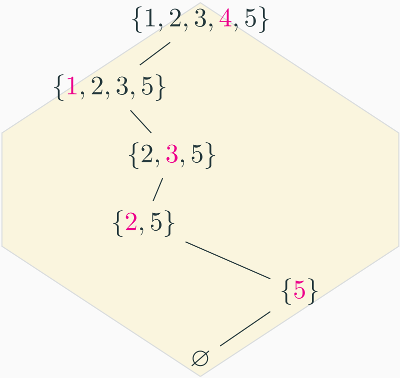

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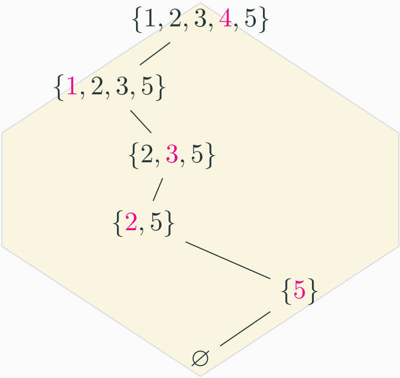
M_f^S has mons in x_S and $x_{\overline{S}}$ as rows and columns.

Observation. $\mathrm{ROABPwidth}_\sigma(f(x_1, \ldots, x_n)) \leq w$ **iff** 'σ traces an $\varnothing$ to $[n]$ path in the graph $H_w(f)$'.

> $H_w(f)$: induced subgraph of hypercube, where $S \in H_w(f)$ if and only if $\mathrm{rk}(M_f^S) \leq w$.



E-time algorithm: Search on the Hypercube

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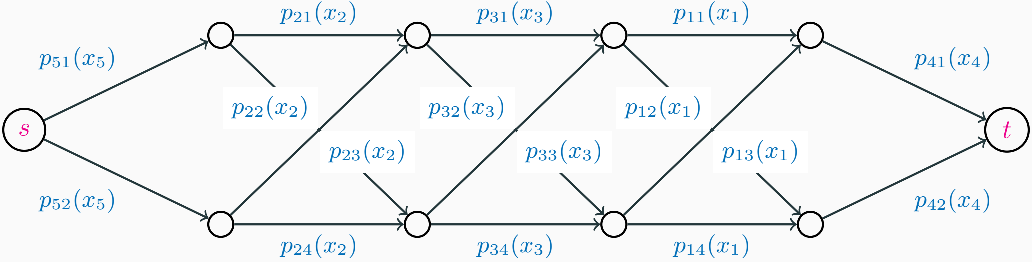
Algorithm. `FindOrder(f,w)`

1. `PopulateGraph(f,w)`: Find $H_w(f)$ using a DFS starting at $\varnothing$ (and above fact).
2. Output any σ that traces an $\varnothing$ to $[n]$ path in $H_w(f)$.

Proof Ideas

Algorithm for the generic case

Random/generic ROABPs



Generic ROABP for $n = 5$, $w = 2$ and $\sigma = (5, 2, 3, 1, 4)$: random coeffs for p_{ij}s.

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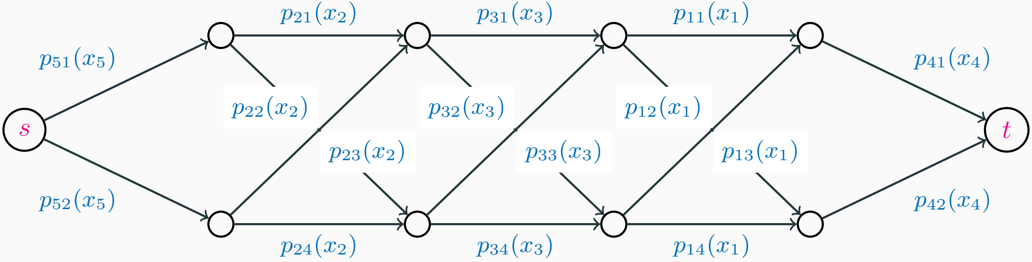

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Definition **($(n, d, w, \sigma, \mathcal{D})$-Generic ROABP)**

ROABP in order σ with all coefficients of edge labels ($\sim ndw^2$) iid according to $\mathcal{D}$.

Bad inputs for `PopulateGraph`

Algorithm. `FindOrder(f,w)`

1. `PopulateGraph(f,w)`: Find $H_w(f)$ using a DFS starting at $\varnothing$ (and above fact).
2. Output any σ that traces an $\varnothing$ to $[n]$ path in the populated graph.

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- We show: for generic $f \in \text{ROABP}(n,d,w,\sigma)$, $H_w(f)$ only has the obvious vertices.
  $\mathrm{rk}(M_f^S) \leq w$ when S is a prefix of σ, or $|S|$ is too small (M_f^S is skewed).

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- We show: for generic $f \in \text{ROABP}(n, d, w, \sigma)$, $H_w(f)$ only has the obvious vertices.
  $\text{rk}(M_f^S) \leq w$ when S is a prefix of σ, or $|S|$ is too small (M_f^S is skewed).
- For "inconsistent" S, fs with $\text{rk}(M_f^S) \leq w$ form a strict subvariety of $\text{ROABP}(n, d, w, \sigma)$.
  [SZ lemma]: $H_w(f)$ has $n^{O(\log_d(w))}$ vertices w.h.p., for any large-enough domain.

Theorem (Average-case algorithm)

Over all sets D of size 2^{10n},and for any n, d, w, σ,
`PopulateGraph` runs in randomized time $n^{O(\log_d(w))} \cdot \mathrm{poly}(d, w)$ on a random/generic input from ROABP(n, d, w, σ) w.h.p., where the coeffs are drawn from D.

- Polynomial time when $w = d^{O(1)}$.
- Quasi-polynomial time even when $d = O(1)$.

Average Case Results

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Remark. We need $|D| \sim 2^n$ due to a union bound over all inconsistent S.

Summary

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  (Only other algebraic MCSP results are for tensor and Waring ranks [Håstad’90].)
- Order-finding can be solved in average case in (quasi-)polynomial time.
- **Approximation algorithms.**
  - ROABPwidth is hard to approximate up to any constant factor under SSE conjecture.
  - Unconditionally, any constant approximation for ROABPwidth leads to a PTAS.

Open Questions

- **Average-case algorithm.**
  - Polynomial time for constant individual degree?

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- **Hardness of approximation.**
  - Is `CutWidth` hard to approximate up to a constant factor (without SSE)?
  - Is ROABPwidth hard to approximate (for some other reason)?

Thank you!

ABP-figure credits: Prerona Chatterjee