Is there an Algebraic Natural Proofs Barrier?

Anamay Tengse

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STCS Symposium

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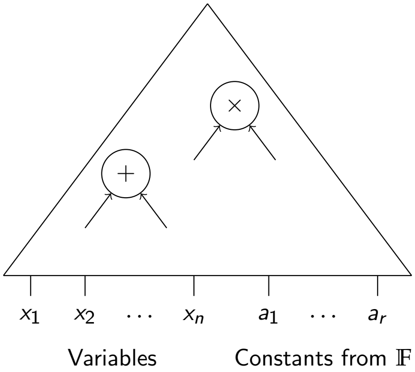
Computing Polynomials

Computing Polynomials

$x_1 \quad x_2 \quad \cdots \quad x_n \qquad a_1 \quad \cdots \quad a_r$

Variables — Constants from $\mathbb{F}$

Computing Polynomials



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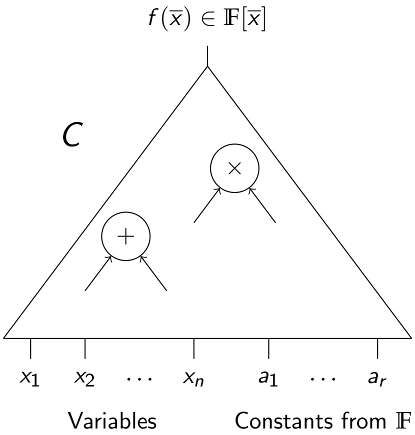

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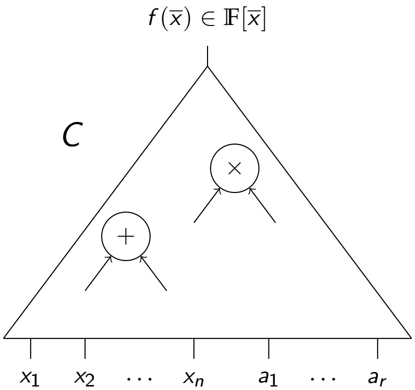
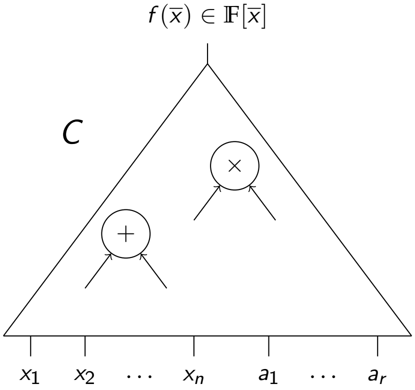


Algebraic Circuit for $f(\overline{x})$

Computing Polynomials



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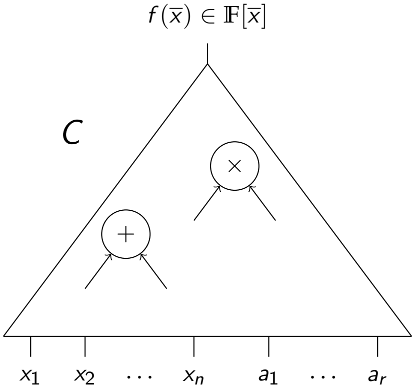
Size(C): Number of *gates*

- Operations used by C

Computing Polynomials



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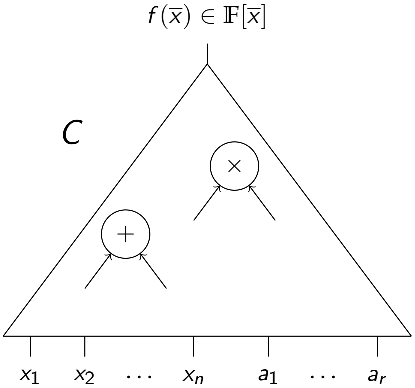

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For this talk.

Variables: n, Degree: d,

Polynomials with $d = \text{poly}(n)$.

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Question. Is VP $=$ VNP?

Some known hardness results

- Hardness Results for Structured Models:
  - Homogeneous constant depth formulas (exponential hardness) [NW95,GKKS13,KS14,...]
  - Multilinear formulas (quasipolynomial hardness) [Raz09,DMPY12,...]
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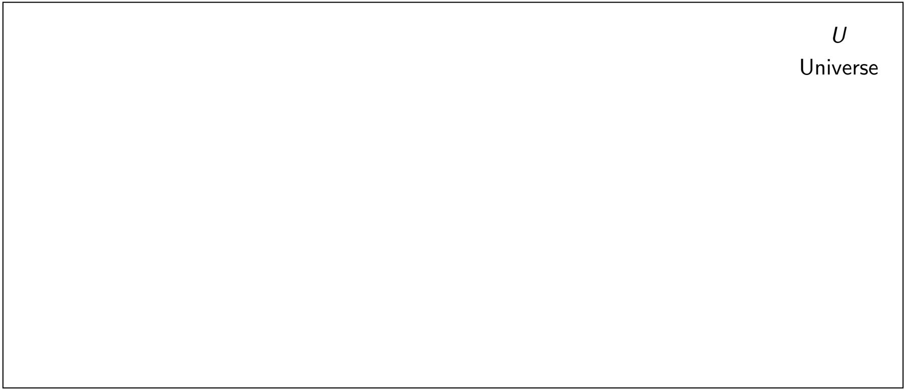
Question. Are the *"natural techniques"* insufficient?

What are "natural techniques"?



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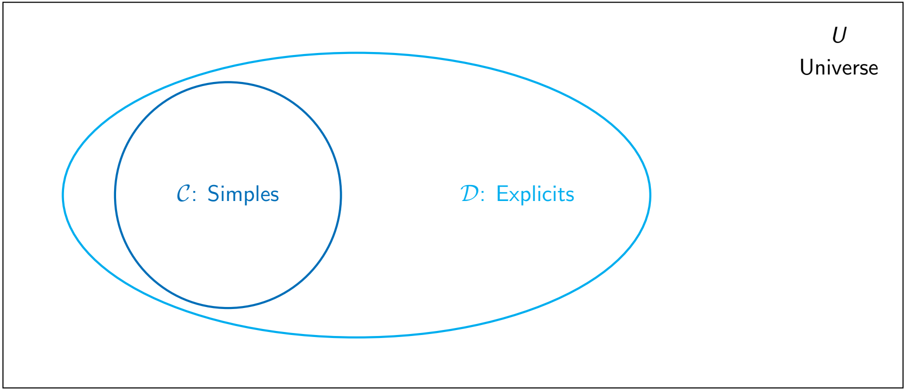

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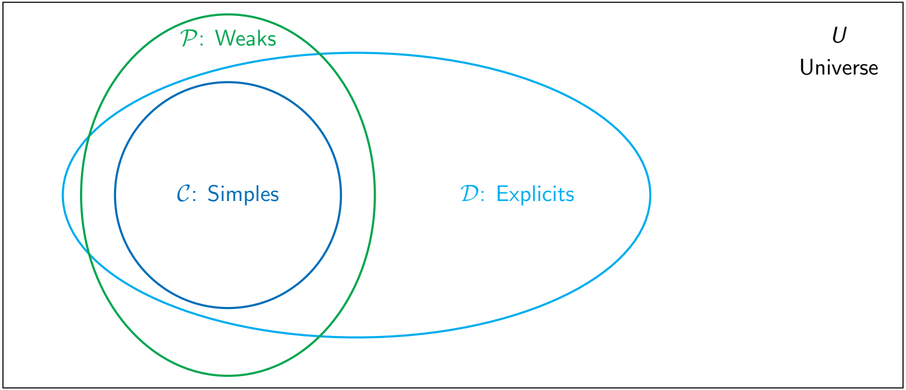

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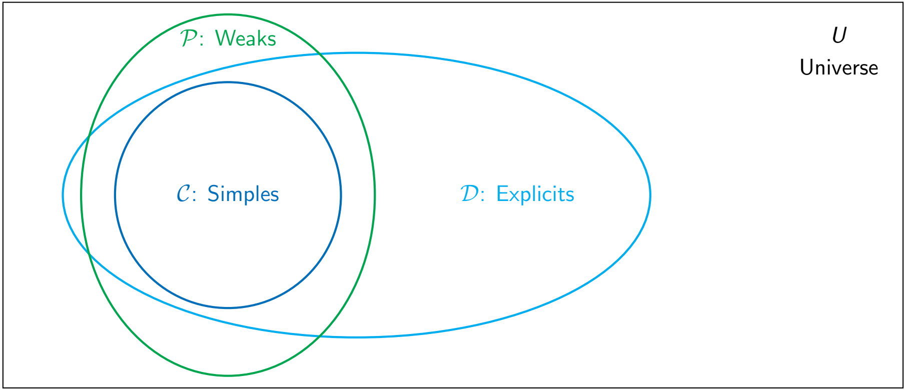

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Easy weakness $\Rightarrow$ Something explicit should be non-weak.

Natural techniques in action

Warm-up

$U = \mathbb{C}^4$

- U - univariates of deg ≤ 3 over $\mathbb{C}$.

Natural techniques in action

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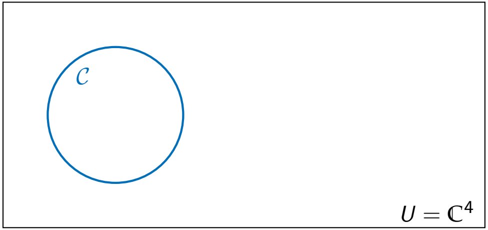

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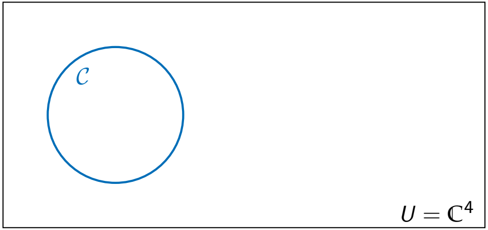

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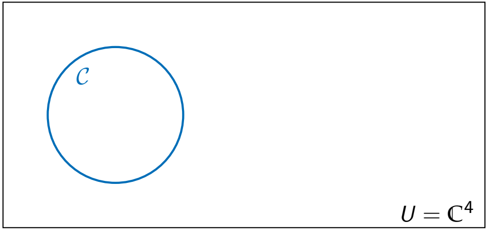

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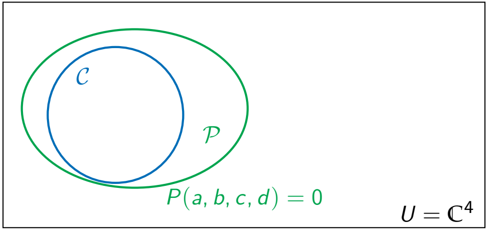

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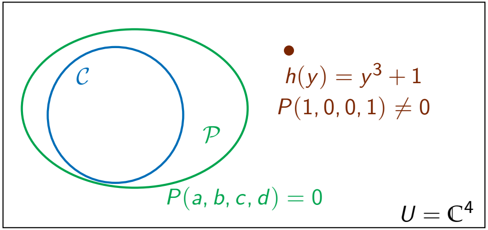

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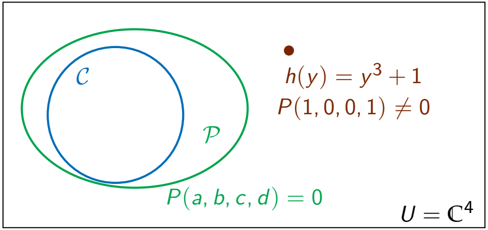

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[FSV18,GKSS17]

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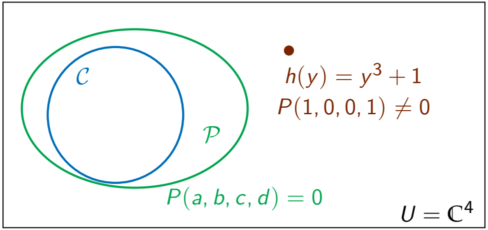

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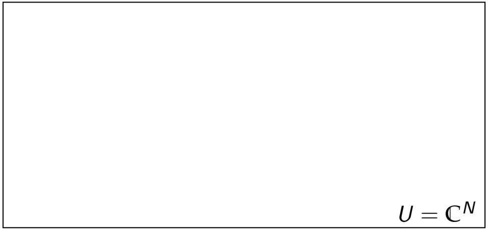

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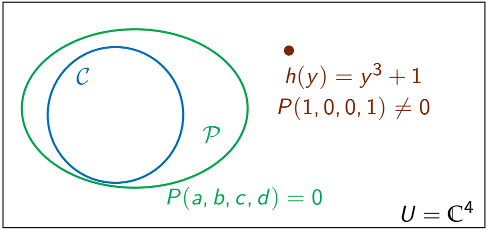

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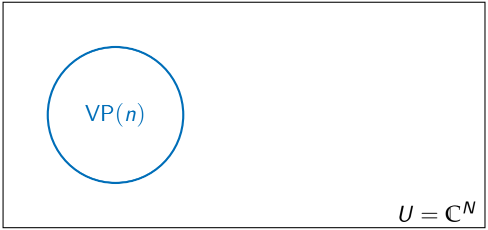

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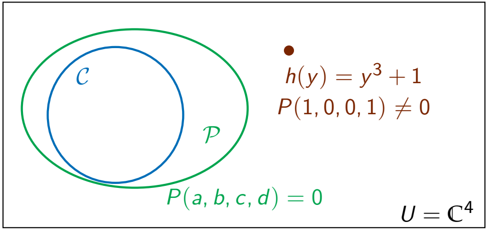

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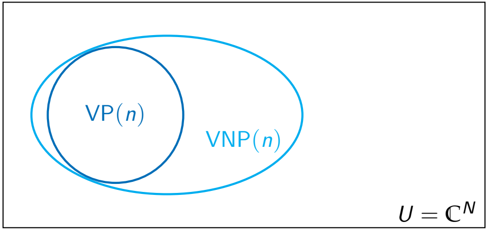

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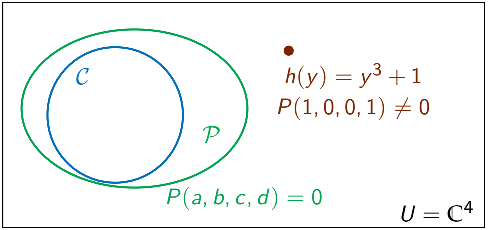

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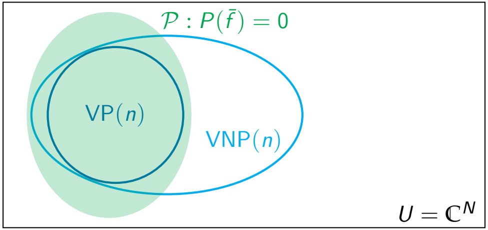

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- $\mathcal{C} = \mathrm{VP}(n)$.
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- **Is there a "simple"** $P(Z_1, \ldots, Z_N)$ **s.t.**

  $P(\bar{f}) = 0$ **for all** $f \in \mathrm{VP}(n)$**?**

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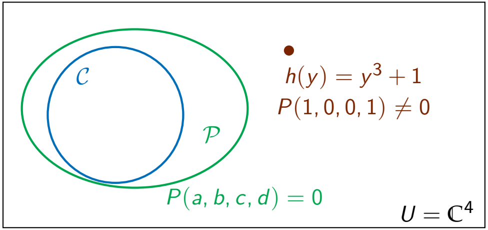

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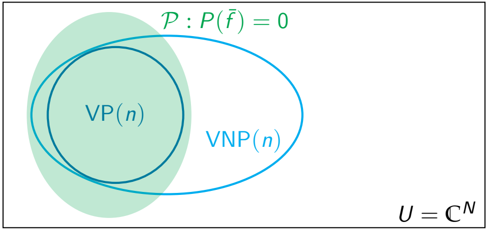

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Barrier: No $P \in \mathrm{VP}(N)$ witnesses the separation of $\mathrm{VP}(n)$ and $\mathrm{VNP}(n)$.

$\equiv$ "Natural techniques" cannot prove $\mathrm{VP} \neq \mathrm{VNP}$.

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- Algebraic natural proofs [FSV18,GKSS17]:
  - If $\mathrm{VP}(n)$ has $\mathrm{poly}\log(n)$-*succinct hitting sets*, then no natural proofs for VP.

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Our results

Dream. There is a $P(Z_1, \ldots, Z_N) \in \mathrm{VP}(N)$ s.t.

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Proofs for "interesting" polynomials

Theorem [CKRST'20]

For all large n, d and $N = \binom{n+d}{d}$, there exists a $P(Z_1, \ldots, Z_N)$ s.t.

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Idea. Hitting sets for $\mathcal{C}$ give natural proofs for $\mathcal{C}'$.

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Idea. The Kabanets-Impagliazzo generator [KI04] can be made VNP-succinct.

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- **Undecided.** The natural proofs question for VP seems quite interesting. :)

Thank You

Webpage: anamay.bitbucket.io

Formal statement of [CKRST'20]

$\exists$ a collection $\mathcal{P}$ of proof families such that,

$\forall$ degree functions $d(n) = \text{poly}(n)$,

the proof family $\left\{P_{N(n)}\right\} = \mathcal{P}(d(n))$ is of $N(n) = \binom{n+d(n)}{n}$ variate polynomials,

and $\forall$ size functions $s(n) = \text{poly}(n)$, $\exists n_0$ such that $\forall n > n_0$,

the polynomial $P_{N(n)}$ vanishes on $\text{Ckt}'(n, d(n), s(n))$.

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The polynomial Perm_m requires size $> 2^{m^{\epsilon}}$, for infinitely many m.

$\exists$ a collection of families of polynomials $\mathcal{H} \subseteq \mathrm{VNP}(n^c)$, such that

the collection $\mathcal{H}(n)$ is a hitting set for VP_N where $N = \binom{n+n^c}{n}$.

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for all degree and size functions $d(N), s(N) = \mathrm{poly}(N)$, there exists an m_0, such that

if for some $m > m_0$, Perm_m requires size $> 2^{m^\epsilon}$,

then for $n(m) = \mathrm{poly}(m), d = n^c$, the collection of polynomials $H_{n(m)} \subseteq \mathrm{VNP}_{n(m)}(n^c)$

is a *hitting set* for the collection $\mathrm{VP}_N(d(N), s(N))$ for $N(n) = \binom{n+n^c}{n}$.