Natural Proofs in Algebraic Circuit Complexity

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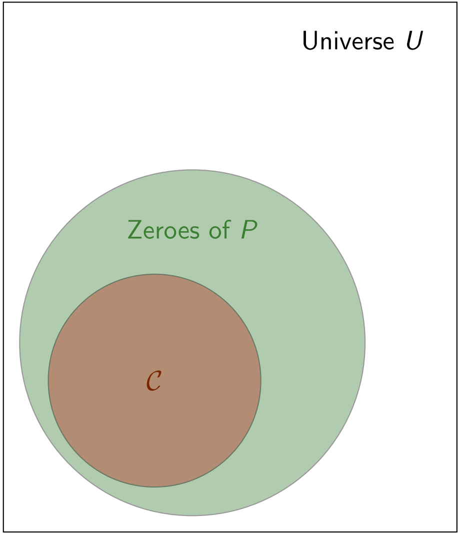
Equations for polynomials



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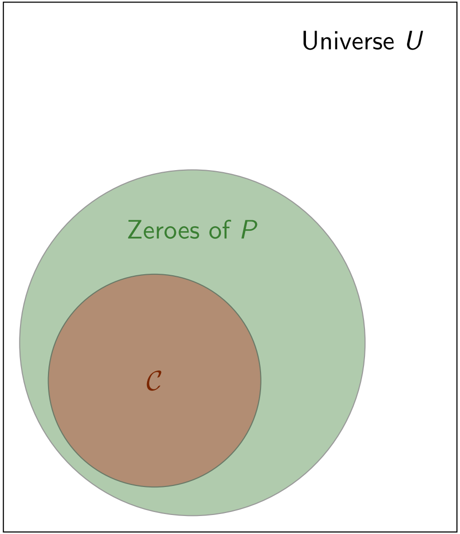

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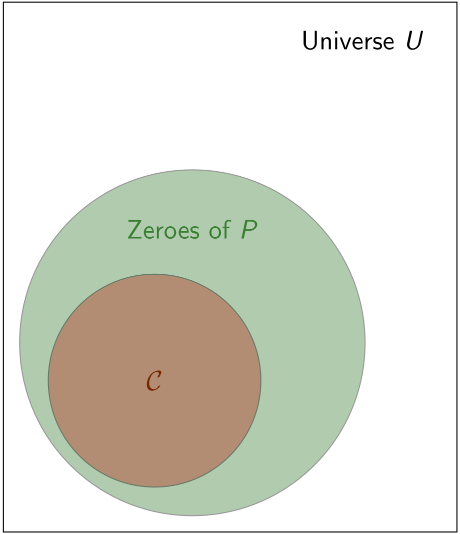

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  Yes, for "explicit" lower bounds

Algebraic Circuits

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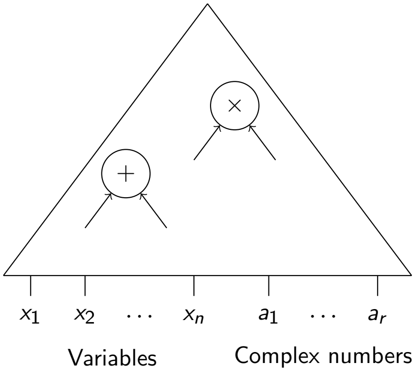
$x_1 \quad x_2 \quad \cdots \quad x_n \qquad a_1 \quad \cdots \quad a_r$

Variables Complex numbers

Algebraic Circuits



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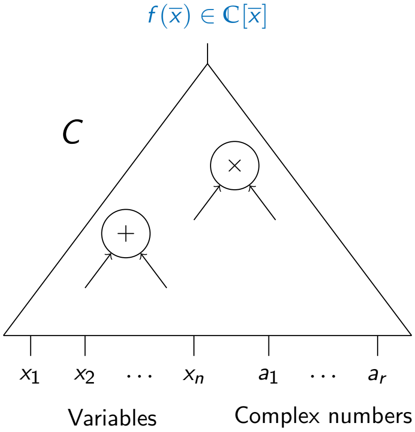

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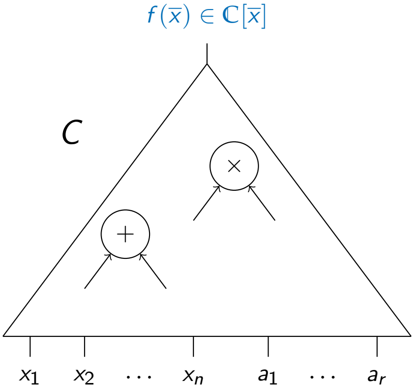

Algebraic Circuit for $f(\overline{x})$

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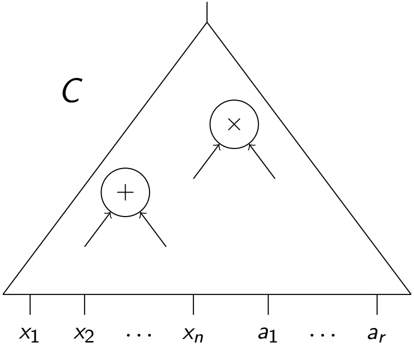
$f(\overline{x}) \in \mathbb{C}[\overline{x}]$



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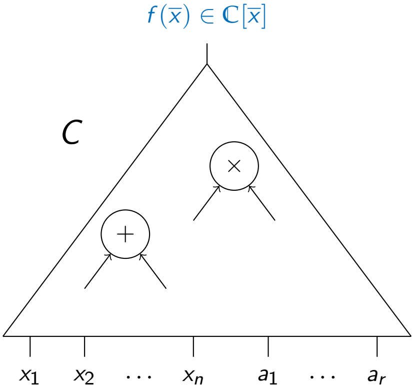

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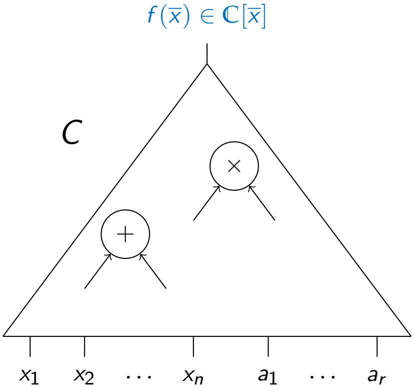

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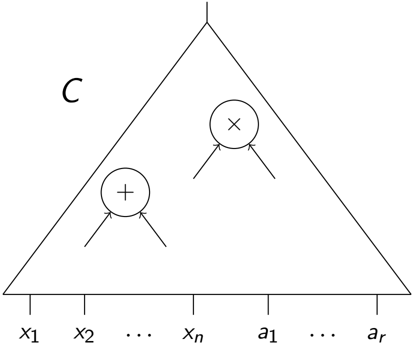

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"Low-degree" polynomials.

Variables: n, Degree: d,

Polynomials with $d = \text{poly}(n)$.

Algebraic circuit complexity: Basics

Boolean world

Algebraic world

- P (or P/ poly)
  — E.g. MaxFlow,Matching
- NP (or NP/ poly)
  — 'verifiable' in poly-time
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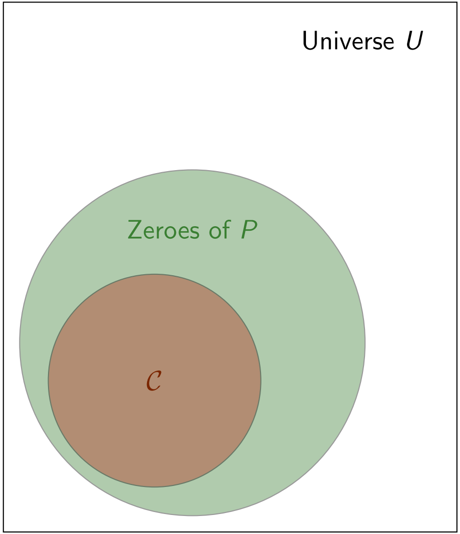
Big questions: VP vs VNP, Det_n vs Perm_n

Equations for Polynomials: Recap



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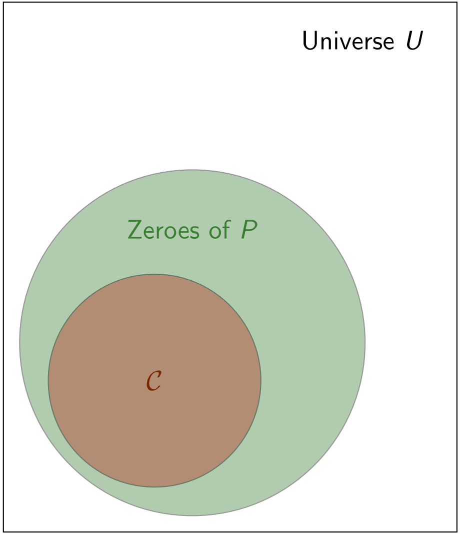

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Rest of this talk:

Assume degree d = number of variables n.

N = Number of coefficients = $\binom{n+d}{d}$,

$N = 2^{O(n)}$.

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Q. Are there (poly-sized) equations for general classes?

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Q. Are there VP-natural proofs for general classes like VP?

Two (types of) Questions

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   — [CKRST20,KRST21]: Bounds on $\mathcal{C}$ using 'hardness-randomness connections'.

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Equations for $\mathcal{C}$ in VP

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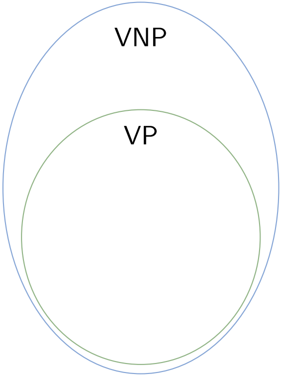
Equations for VP in $\mathcal{D}$



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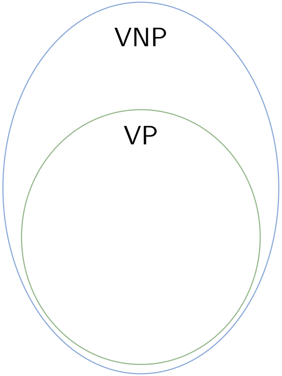

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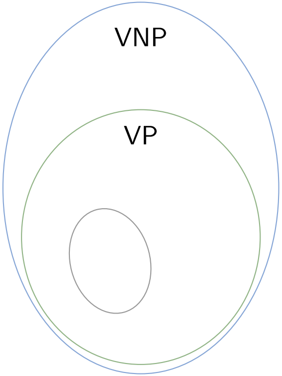

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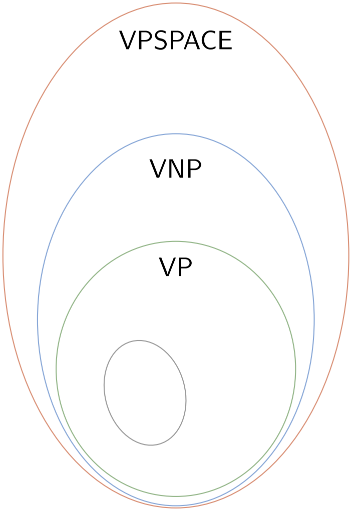

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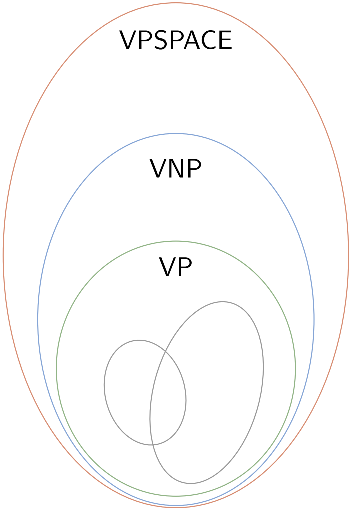

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[CKRST20]: VP$' \subseteq \mathcal{C}$

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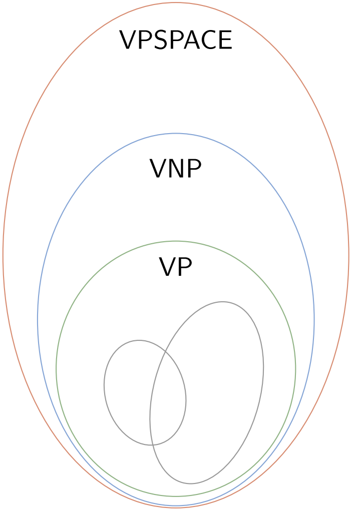

[KRST21]: VNP $\not\subseteq \mathcal{C}$

If Perm is $\exp(n^{\epsilon})$-hard

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Equations for $\mathcal{C}$ in VP

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- Key Idea: Equations from hitting sets.

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- Defn (Hitting Set) Set $S \subset \mathbb{Z}^n$ is a *hitting set* for a class $\mathcal{C}$, if for all $f \in \mathcal{C}$ there is some $a \in S$: $f(a) \neq 0$.
- "Diagonalisation using hitting sets" [HS80,Agr05]: If $h(a) = 0$ for all $a \in S$, then $h \notin \mathcal{C}$.
- [HS80]: There *exist* efficient hitting sets S, for VP.

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- Efficient hitting sets also exist for VNP,
  [?!] The same result holds for the analogous class VNP'.
  — BUT if some $h \in \mathsf{VNP}'$ (say Perm) vanishes on a hitting set for VP,
  then that hitting set gives a "VP-natural proof for $\mathsf{VP} \neq \mathsf{VNP}$"!!

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Hardness $\longrightarrow$ Psuedorandomness
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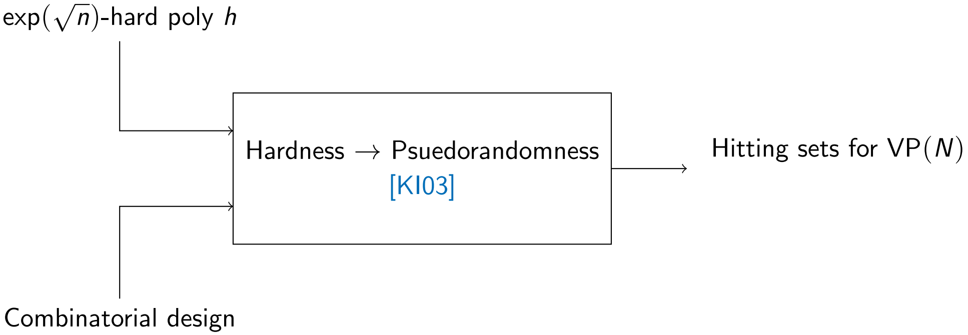

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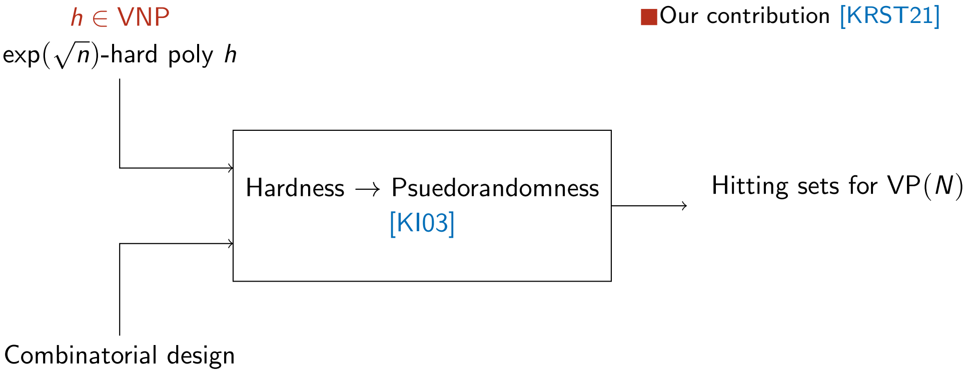

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- "Efficient equations give explicit lower bounds".

  Subject to Perm being $2^{n^{\epsilon}}$ hard.

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- "Natural methods are sufficient"
  - (Conditionally) extend [CKRST20] to work for VP with coefficients of size $2^{n^{\omega(1)}}$ ...?
  - Due to [KRST21], equations for coefficients of size $2^{\mathrm{poly}(n)}$ would essentially guarantee a "natural separation" of VP and VNP.

Thank You

Questions?