Near-optimal Bootstrapping of Hitting Sets

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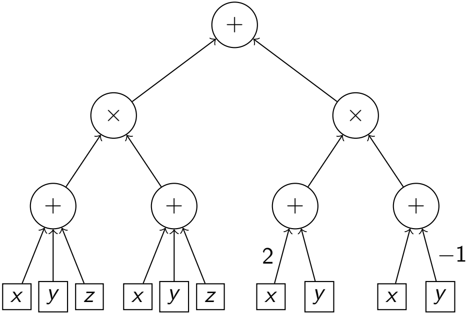
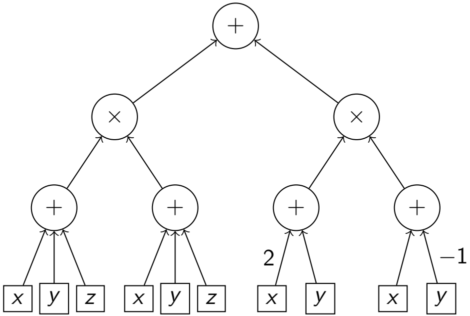
[1]Partially supported by **Microsoft Research**, via SIAM travel awards.

Algebraic Models



Algebraic Formula

Algebraic Models



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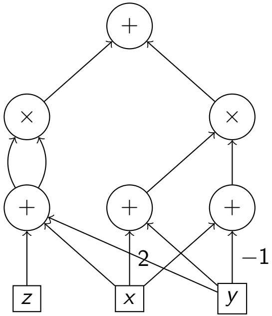

Algebraic Circuit

The Hardness Question

$$f = \mathrm{Det}\left(\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix}\right)$$

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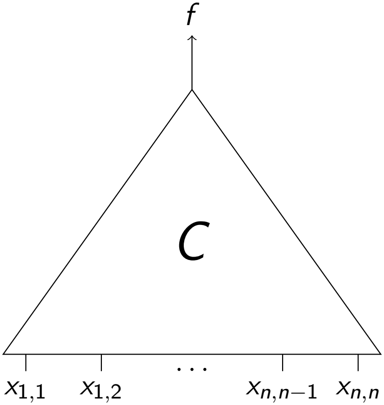

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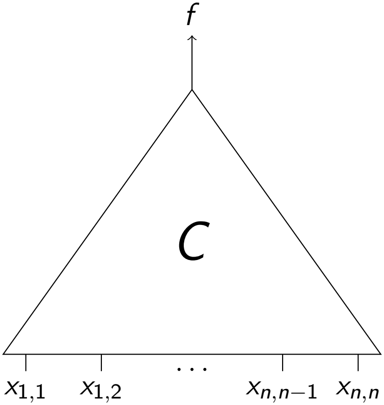

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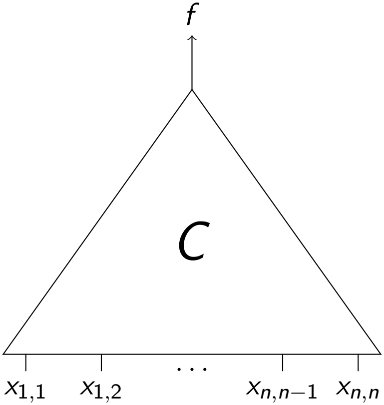

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Find an "explicit" n-variate $f(\mathbf{x})$ that requires $n^{\Omega(1)}$ sized circuits?

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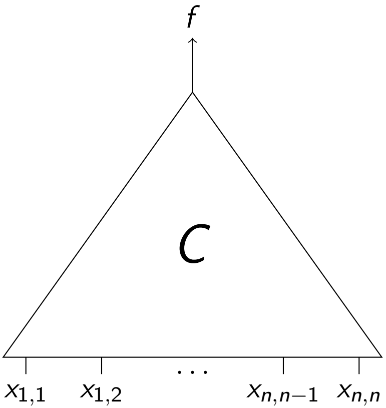

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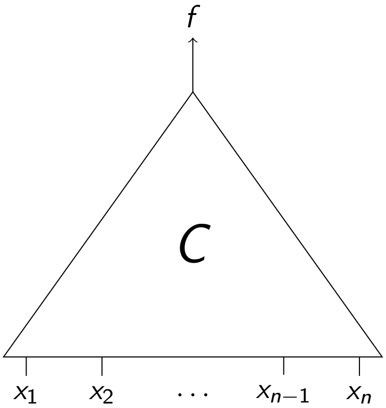
Find an "explicit" n-variate $f(\mathbf{x})$ that is hard for circuits? **OPEN**

Identity Testing



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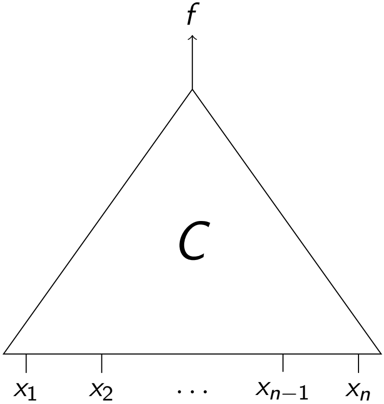

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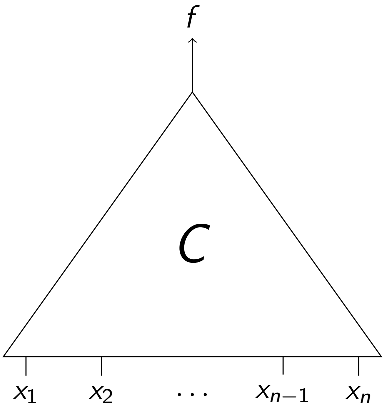

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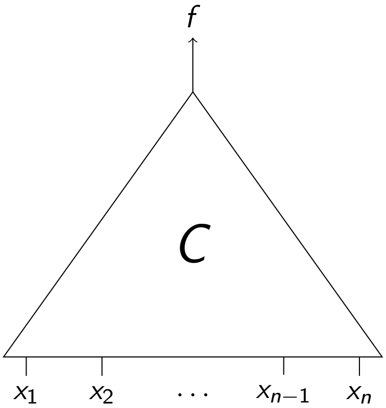

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Whitebox: Does the given circuit compute 0?

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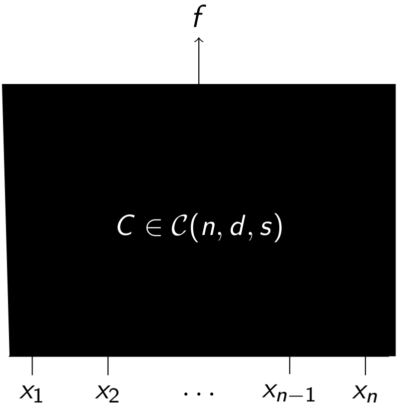

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Blackbox: Look at evaluations of C to tell if $C = 0$.

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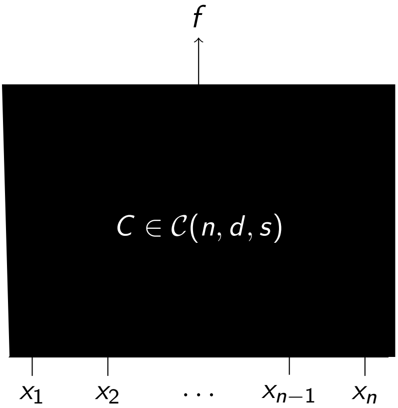

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Hitting Set: Find $H_{\mathcal{C}}$ such that $C = 0$ **iff** it is 0 on every $h \in H_{\mathcal{C}}$.

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OPEN: Find an explicit hitting set of size $d^{o(n)}$ for $\mathcal{C}(n, d, s)$.

Improving slightly non-trivial Hitting Sets

Theorem [Agrawal, Ghosh, Saxena 2018]
Suppose for a large constant n and all $s \geq n$, there is an explicit hitting set of size

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$\mathrm{tiny}(s) = \exp(\exp(O(\log^* s)))$

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Q. Why $n^{0.5}$? What about $s^{o(n)}$?

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Theorem [Kumar, Saptharishi, T]
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Hitting Set

Hard Polynomial

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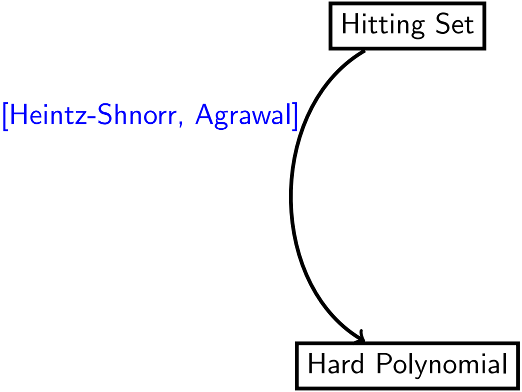

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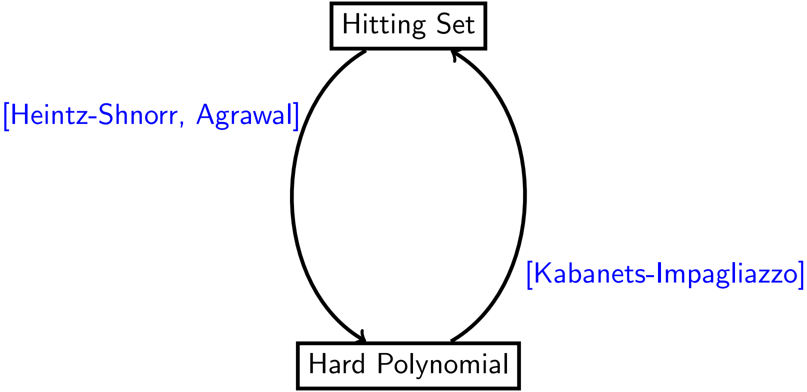

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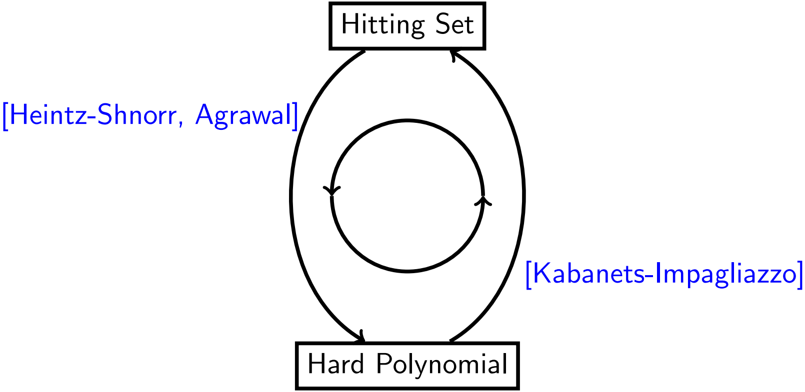

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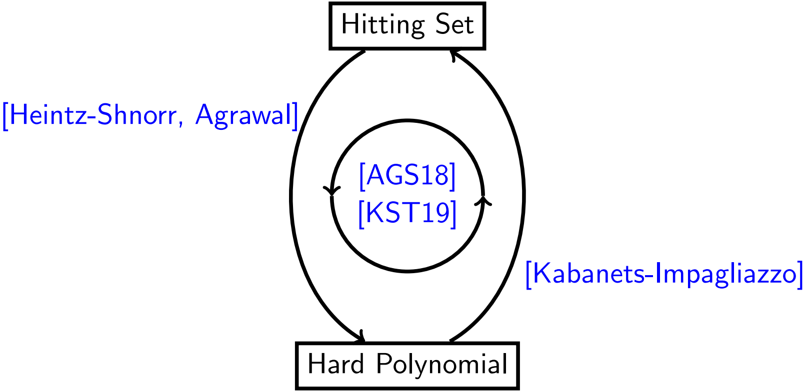

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Proof Idea: Use interpolation to get a Q_k that vanishes on H.

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when $\mathbf{y}_1, \ldots, \mathbf{y}_m$ *are nearly disjoint.*

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Outcome: $\mathrm{PIT}(m, d, s)$ reduces to $\mathrm{PIT}(k^2, d', s')$,

for slightly larger d', s' and $k \sim \mathrm{polylog}(m)$.

Template for Bootstrapping

Hyp.: Q_k of ind. deg. $< d$ hard for $\mathcal{C}(k^2, s^{15}, s^{15})$.

Goal: Hitting set for $\mathcal{C}(m, s, s)$, for $m = 2^{\sqrt{k}}$, all s.

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A. We get a $s^{50g(k)^2}$ hitting set for $\mathcal{C}(2^{\sqrt{k}}, s, s)$, for all s.

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Unlike the boolean case, nothing stops us.

High Level Overview

Bootstrapping

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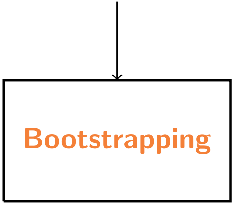
$S^{g_1(k)}$ for $\mathcal{C}(k^2, S, S)$



High Level Overview

$S^{g_1(k)}$ for $\mathcal{C}(k^2, S, S)$

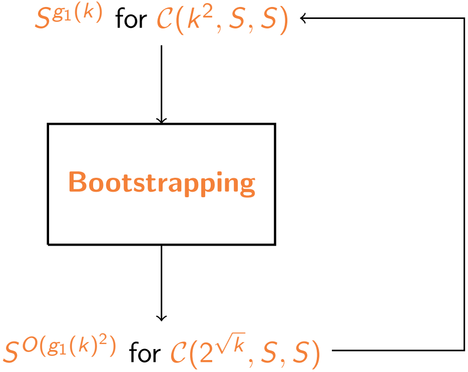
↓

Bootstrapping

↓

$S^{O(g_1(k)^2)}$ for $\mathcal{C}(2^{\sqrt{k}}, S, S)$

High Level Overview



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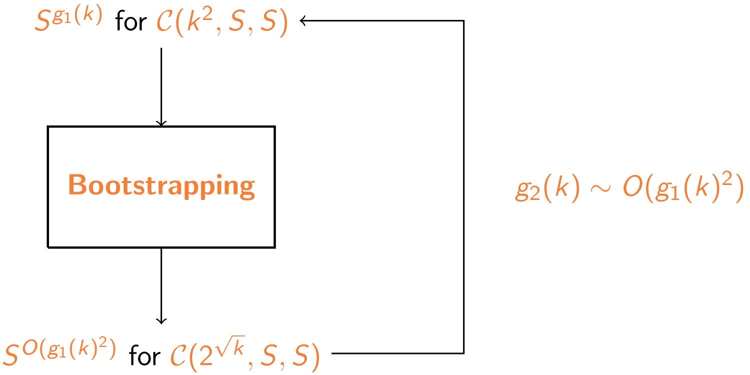

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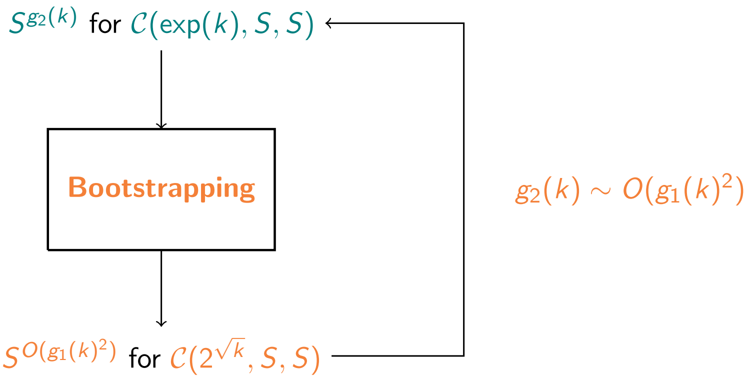

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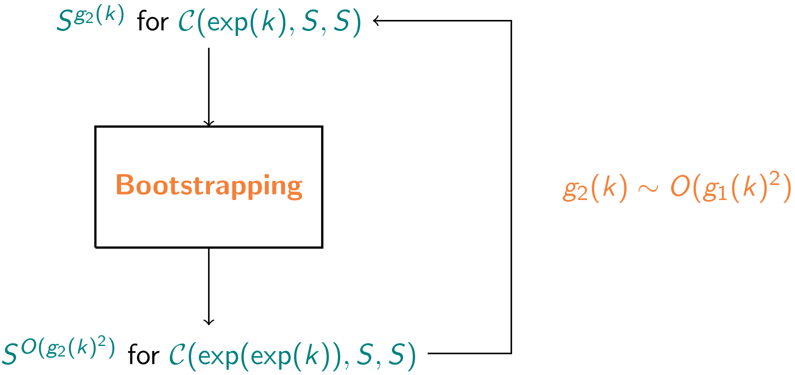

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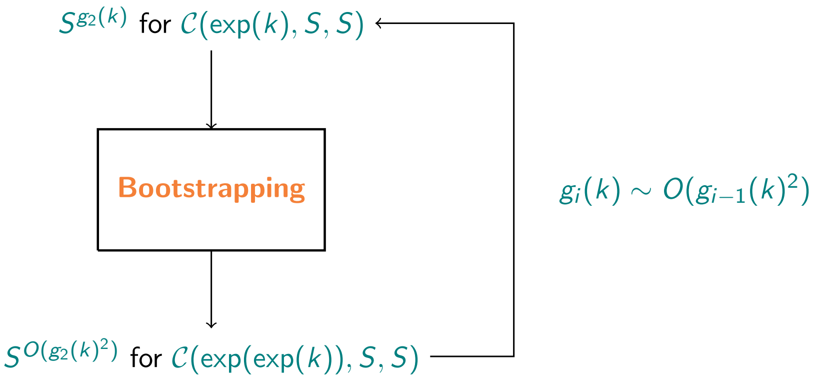

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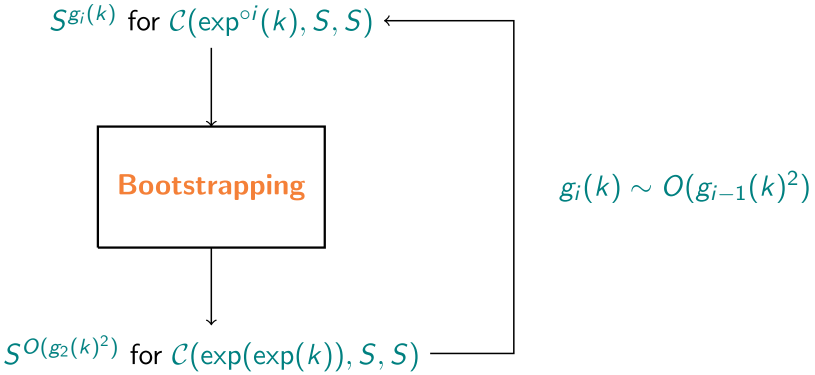

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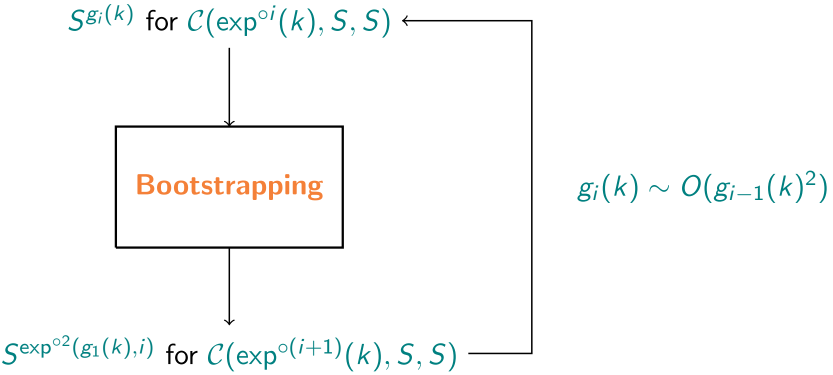

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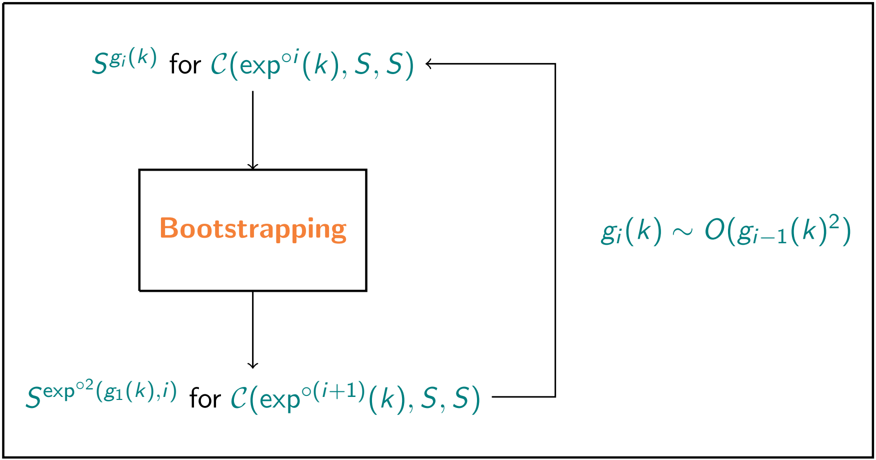
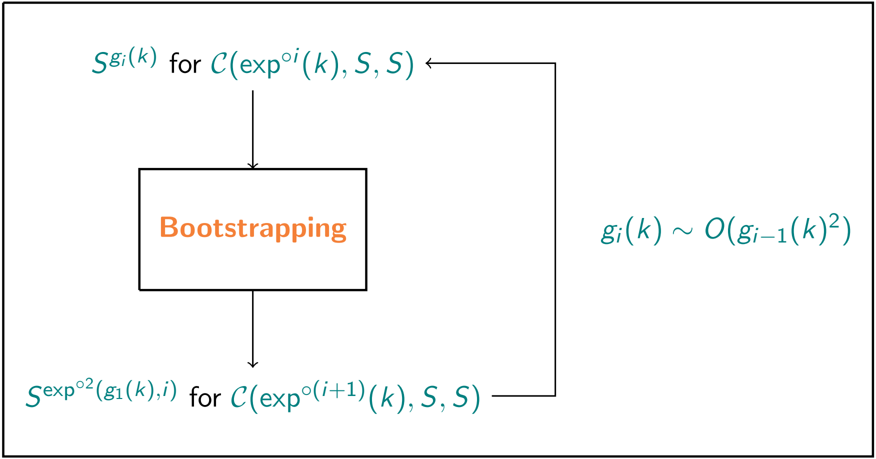


High Level Overview

$$S^{n^{0.2}} \text{ for } \mathcal{C}(n, S, S) \longrightarrow S^{\exp(\exp(O(\log^* S)))} \text{ for } \mathcal{C}(S, S, S)$$



Summary and Open Questions

Theorem [Kumar, Saptharishi, T]
Suppose for a large constant n and all $s \geq n$, there is an explicit hitting set of size

$$s^{o(n)} \quad \text{for} \quad \mathcal{C}(n, s, s).$$

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Thank You!